Final exam Linear Algebra II

Thursday 04/04/2024, 15:00–17:00, Martini Plaza

1 (9 = 3 + 2 + 4 pts) **Subspaces and bases**

Let $\mathcal{V}$ be the $\mathbb{R}$-vector space of all functions $f : \mathbb{N} \to \mathbb{R}^n$, where $\mathbb{N} = \{1, 2, \ldots\}$. Here the addition of functions $f, g \in \mathcal{V}$ is defined as $(f + g)(k) = f(k) + g(k)$ for all $k \in \mathbb{N}$, while scalar multiplication is defined as $(af)(k) = af(k)$ for $a \in \mathbb{R}$ and $f \in \mathcal{V}$. In this exercise we will consider the set

$$\mathcal{S} := \{f \in \mathcal{V} \mid f(k+1) = T(f(k)) \ \forall k \in \mathbb{N}\},$$

where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator.

(a) Prove that $\mathcal{S}$ is a subspace of $\mathcal{V}$.

(b) Let $x \in \mathbb{R}^n$. Show that f, defined by $f(k) = T^{k-1}(x)$ for all $k \in \mathbb{N}$, is in $\mathcal{S}$.

(c) Find a basis for $\mathcal{S}$. Motivate your answer.

2 (9 = 2 + 2 + 2 + 3 pts) **Inner product, injective, bijective, adjoint**

Let V and W be two inner product spaces over $\mathbb{C}$. The inner product on V we denote as $\langle \cdot, \cdot \rangle_V$ and the one on W as $\langle \cdot, \cdot \rangle_W$. We call a $\mathbb{C}$-linear transformation $T \colon V \to W$ *unitary* if $\langle v_1, v_2 \rangle_V = \langle T(v_1), T(v_2) \rangle_W$ for all $v_1, v_2 \in V$.

(a) Taking $V = \mathbb{C}^2$ and $W = \mathbb{C}^3$ (both with the standard Hermitian inner product), show that $T \colon V \to W$ given by $\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ w \\ 0 \end{pmatrix}$ is unitary but not surjective.

(b) Prove that for every V, W as above, every unitary $T \colon V \to W$ is injective.

(c) Now suppose that moreover $\dim V = \dim W < \infty$ and that $T \colon V \to W$ is unitary. Prove that T is bijective.

(d) In the situation of (c), show that T^{-1} is the adjoint of T.

3 $(9 = 3 + 3 + 3 \text{ pts})$ **Inner product, norm**

Consider a $\mathbb{C}$-inner product space V of dimension 2 over $\mathbb{C}$, and two independent vectors $v_1, v_2 \in V$. The goal of this exercise is to describe all possible inner products on V.

(a) Suppose that $\langle \cdot, \cdot \rangle$ is an inner product on V. Put $a := \langle v_1, v_1 \rangle$ and $b := \langle v_2, v_2 \rangle$ and $c := \langle v_1, v_2 \rangle$. For an arbitrary pair $v = \alpha v_1 + \beta v_2$, $w = \gamma v_1 + \delta v_2$ in V (with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$), write $\langle v, w \rangle$ in terms of $a, b, c, \alpha, \beta, \gamma, \delta$.

(b) Now take $a, b, c \in \mathbb{C}$. Show that if

$$\begin{cases} (v_1, v_1) \mapsto a, \\ (v_1, v_2) \mapsto c, \\ (v_2, v_2) \mapsto b \end{cases}$$

extends to an inner product $V \times V \longrightarrow \mathbb{C}$, then $a, b \in \mathbb{R}_{>0}$ and $|\mathrm{Re}(c)| < \sqrt{ab}$.

(c) Vice versa, prove that if $a > 0$ and $b > 0$ and $c \in \mathbb{C}$ satisfy $|\mathrm{Re}(c)| < \sqrt{ab}$, then

$$\begin{cases} (v_1, v_1) \mapsto a, \\ (v_1, v_2) \mapsto c, \\ (v_2, v_2) \mapsto b \end{cases}$$

extends to an inner product $V \times V \longrightarrow \mathbb{C}$.

4 $(9 = 2 + 3 + 4 \text{ pts})$ **Positive semidefinite matrices**

Let $\mathcal{X}$ be a nonempty set. The function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is called *positive semidefinite* if its associated $n \times n$ *Gram matrix* $(k(x_i, x_j))$ is symmetric and positive semidefinite for all $x_1, x_2, \ldots, x_n \in \mathcal{X}$ and all $n \in \mathbb{N}$.

(a) Suppose that $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is positive semidefinite and consider $x_1, x_2, \ldots, x_n \in \mathcal{X}$. Show that all eigenvalues of the Gram matrix $(k(x_i, x_j))$ are nonnegative.

(b) Let $\mathcal{V}$ be an inner product space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle$. Consider a function $f : \mathcal{X} \to \mathcal{V}$. Define $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ by $k(x, y) := \langle f(x), f(y) \rangle$. Prove that k is positive semidefinite.

(c) Without computing eigenvalues or determinants, show that the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 9 & 16 \\ 1 & 9 & 25 & 49 \\ 1 & 16 & 49 & 100 \end{bmatrix}$$

is positive semidefinite. *Hint:* use $f \colon \mathbb{R} \to \mathbb{R}^3$ given by $f(x) = \begin{pmatrix} 1 \\ \sqrt{2}x \\ x^2 \end{pmatrix}$ and (b) and $(x_1, x_2, x_3, x_4) = (0, 1, 2, 3)$.